Peter Borwein: A philosopher's mathematician

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1 Early years

In 1984, a relatively obscure mathematician and computer scientist working at NASA's Ames Research Center in California noticed an interesting and unusual article in the latest edition of *SIAM Review* [2]. Written by Jonathan Borwein and Peter Borwein, it presented new "quadratically convergent" algorithms for π, $\log 2$ and various transcendental functions (sin, cos, exp, log, etc). The article defined "quadratically convergent" to mean that each iteration approximately *doubles* the number of correct digits, provided, of course, that all computations are performed to at least the level of numeric precision desired for the final result.

Intrigued, this mathematician immediately set out to write computer programs to implement these algorithms, including the requisite software to perform calculations to an arbitrary level of precision. After some effort, he succeeded to calculate π, for instance, to roughly one million decimal digits. He contacted Peter Borwein, then at Dalhousie University in Canada, who in turn referred him to his brother Jonathan Borwein. The two Borweins then sent this mathematician some new results, including the following algorithm: Set $a_0 = 6 - 4\sqrt{2}$ and $y_0 = \sqrt{2} - 1$. Then iterate, for $k \geq 0$,

$$\begin{aligned} y_{k+1} &= \frac{1 - (1 - y_k^4)^{1/4}}{1 + (1 - y_k^4)^{1/4}} \\ a_{k+1} &= a_k (1 + y_{k+1})^4 - 2^{2k+3} y_{k+1} (1 + y_{k+1} + y_{k+1}^2). \end{aligned} \tag{1}$$

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Then $1/a_k$ converge *quartically* to π: each iteration approximately *quadruples* the number of correct digits (presuming as before that each iteration is performed using a level of numeric precision at least as great as desired for the final result). This mathematician immediately set out to implement this algorithm on one of NASA's supercomputers. Soon he succeeded in computing π to 29,300,000 digits, which at the time was the most ever computed.

Later this mathematician and the two Borwein brothers co-authored an article on the underlying theory, published in the *American Mathematical Monthly*. This paper, titled "Ramanujan, modular equations, and approximations to pi, or How to compute one billion digits of pi" [3], was subsequently awarded the Chauvenet and Merten Hasse prizes of the Mathematical Association of America. So began a very productive collaboration spanning four decades, even though they only rarely met in person.

As the reader may have already guessed, this "relatively obscure mathematician" at NASA's Ames Research Center was myself. I was awed at the mathematical brilliance of the Borwein brothers, and more than willing to offer my expertise in computational mathematics and high-performance computing for various research projects suggested by the Borweins.

2 The BBP formula for pi

Although the majority of my subsequent collaboration with the Borwein brothers was with Jonathan, I had the great privilege of one particularly interesting collaboration with Peter. This began in 1995, when Peter Borwein (then the Director of the Centre for Constructive and Experimental Mathematics at Simon Fraser University in Canada) raised the following question: Is it possible to directly calculate one or more digits of some common irrational or transcendental constants, starting at a given position, any faster than simply computing all digits up to and including the given position? In other words, is it possible to peer, as with a telescope, into the high-order decimal or binary expansion of some constant?

Peter quickly deduced a remarkable scheme that did just that for the constant $\log 2 = 0.693147180559945\ldots$. Recall Euler's classical formula

$$\log 2 = \sum_{k=1}^{\infty} \frac{1}{k 2^k}. \tag{2}$$

Given some position d, note that binary digits of $\log 2$ beginning at position $d+1$ can easily be found by calculating frac $\left(2^d \log 2\right)$, where frac means fractional part. This can be written as follows, after splitting the above sum (2) into two parts:

$$\operatorname{frac}\left(2^d \log 2\right) = \operatorname{frac}\left(\sum_{k=1}^{d} \frac{2^{d-k} \bmod k}{k}\right) + \operatorname{frac}\left(\sum_{k=d+1}^{\infty} \frac{2^{d-k}}{k}\right), \qquad (3)$$

where mod k is inserted in the numerator of the first expression since we are only interested in the fractional part after dividing by k. The exponentiation $2^{d-k} \bmod k$ can be rapidly calculated by using the well-known binary algorithm for exponentiation mod k; the second expression can be evaluated simply as stated, since its terms quickly become very small; and both expressions can be summed using ordinary double-precision floating-point arithmetic (although for large calculations quad-precision floating-point must be used to counter round-off error). A computer implementation quickly confirmed that it worked!

This immediately raised the question of whether a similar trick would work for other well-known mathematical constants. What about π, Peter Borwein's favorite number? No such formula was then known for π. Simon Plouffe, who had assisted Peter to this point, suggested that they do a computer search, starting with a list of constants defined by formulas that permitted the above trick, and performing an integer relation search, using code that I provided that implements the PSLQ integer relation algorithm using very high precision arithmetic. If some integer relation was found between these constants and π, then solving that relation for π would yield a formula for π with the requisite property. The search was on!

After a few months of fits and starts, with additional constants added to the list, Simon Plouffe's program eventually found a relation, which, after some algebra, yielded what is now known as the BBP formula for π:

$$\pi = \sum_{k=0}^{\infty} \frac{1}{16^k}\left(\frac{4}{8k+1} - \frac{2}{8k+4} - \frac{1}{8k+5} - \frac{1}{8k+6}\right). \qquad (4)$$

We included, in our joint paper [1], the results of using this formula to calculate hexadecimal (base-16) digits of π starting at position 10 billion. Others soon computed digits at more distant indices. The current record is hexadecimal digits of π starting at position 100 quadrillion, by Daisuke Takahashi, who used a variation of the BBP formula due to Bellard [4].

3 Later life

Shortly after the work on the BBP formula, Peter had some very disheartening news: He had multiple sclerosis, a disease of the insulating sheaths of nerve cells in the brain and spinal cord. While he valiantly continued with his mathematical work, and remained a leader of the Centre for Constructive and Experimental Mathematics at Simon Fraser University for some time, there is only so much one can do with this increasingly debilitating affliction.

I visited Peter at his home in 2019. At this point it was very difficult for him to function; he required nearly constant home health care. Yet he remained cheerful and optimistic, asking about my family and research, and about developments in the field. Even at the end, he was interested in the big picture: how did all of these developments fit together, and what were the prospects for the future of mathematics, computing and science? He died not long afterwards, in August 2020, outliving his brother Jonathan, who tragically died of a heart attack in 2016, by four years. Peter and Jonathan were certainly among the greatest mathematicians of our time. The field dearly misses them.

References

[1] David H. Bailey, Peter B. Borwein and Simon Plouffe, "On the rapid computation of various polylogarithmic constants," *Mathematics of Computation*, vol. 66, no. 218 (Apr. 1997), 903–913.

[2] Jonathan M. Borwein and Peter B. Borwein, "The arithmetic–geometric mean and fast computation of elementary functions," *SIAM Review*, vol. 26, no. 3 (Jul. 1984), 351–366.

[3] Jonathan M. Borwein, Peter B. Borwein and David H. Bailey, "Ramanujan, modular equations, and approximations to pi, or How to compute one billion digits of pi," *American Mathematical Monthly*, vol. 96, no. 3 (Mar. 1989), 201–219.

[4] Daisuke Takahashi, "Computation of the 100 quadrillionth hexadecimal digit of pi on a cluster of Intel Xeon Phi processors," *Parallel Computing*, vol. 75 (2018), 1–10.